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Time in General Relativity

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Abstract

In order to distinguish between physical and coordinate effects in an arbitrary gravitational field, the space coordinate system and the clock rates must be specified operationally *a priori*. Once this is done, it is no longer possible to set up an initial surface arbitrarily, since this operation must be consistent with certain physical experiments, whose results depend upon the particular physical situation. A method is given for setting up the initial surface, and the time evolution of the system is discussed.

1. *Introduction*

In order to be able to distinguish between changes in the gravitational field variables due to physical effects and changes due to coordinate effects (Komar, 1958; Basri, 1965), it is necessary to specify the spatial coordinate system and the type of coordinate clocks used, from the beginning. The process of setting up a spatial coordinate system in an arbitrary gravitational field has been clarified (Basri, 1965, 1966). The time coordinate, on the other hand, presents some difficulties, when the spatial frame is chosen *a priori*. For instance, if two clocks are synchronized with a third clock, whether these two clocks are synchronous with each other or not depends on the particular physical situation.

If we assume that at each spatial point there is a clock, then the locus of events which occur when the readings of all the clocks are identical defines a constant-time surface. The set of all constant-time surfaces constitutes a global time measure in Einstein's general theory of relativity. In the following discussion concerning time and events which occur at the same time, the above description of global time is understood. This must not be confused with the process of synchronization.

We shall restrict the term synchronization to mean Einstein's synchronization convention (Reichenbach, 1958). This convention is shown diagrammatically in Fig. 1; in this and subsequent figures, the following conventions will be adhered to: a non-vertical solid line is a light signal, with the arrowhead indicating direction; a dashed line indicates a constant global time

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surface; and dashed arrows from a to b indicate that b is synchronous with a. If a light signal leaves particle B at event p, is reflected at event a on particle A, and arrives back at B at event q, then the event on particle B which is synchronous with event a on A is that event which occurs midway in time between the departure and arrival of the light signals. That is, if

$$t_b = t_p + \tfrac{1}{2}t_B(p,q), \qquad t_B(p,q) = (t_q - t_p)_B \tag{1.1}$$

then b is synchronous with a. The event on B which occurs at the same global time as event a is the event a', which is in general different than b.

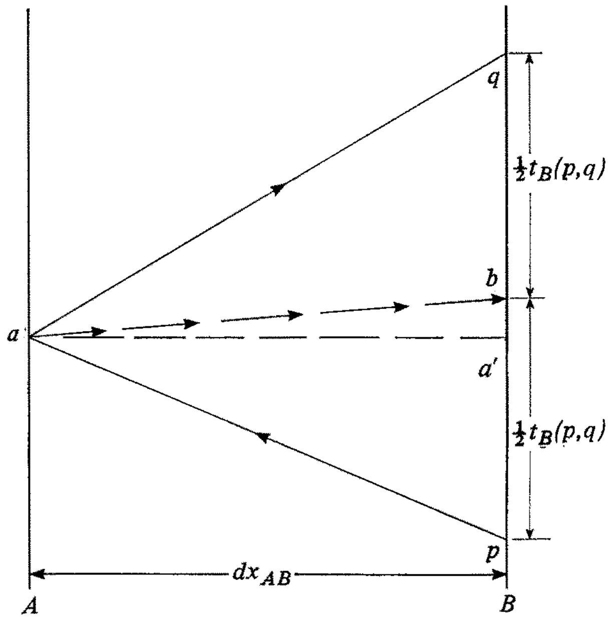

Figure 1—Global and synchronized times.

Thus events on a constant-time surface are not necessarily synchronous. In any case, a 'synchronized time' surface may not be possible to establish (Landau & Lifshitz, 1962, p. 275), because the synchronization process may be neither symmetric nor transitive; moreover, synchronization is only defined for neighboring clocks and hence may not be applied over a finite interval. It is not a defect in the theory that neighboring events on a constant-time surface may not be synchronous, as long as the time parameter is adequate to describe measurable quantities, some of which are discussed in the next section.

The purpose of this paper is to discuss the difficulties associated with global time, and to show how they can be overcome. Einstein's summation convention is used throughout, with Greek indices taking on the values 0, 1, 2, 3, and the Latin indices the values 1, 2, 3.

2. *Pertinent Experiments*

Having chosen the spatial coordinate system and the clocks, we must devise a method of setting the clocks with respect to one another. All the clocks are set to the same initial value on an initial hypersurface which cannot be chosen arbitrarily, but must be established so as to be consistent with physical experiments whose results depend upon the particular physical situation. Four such experiments, all closely related, are the transitivity and symmetry of synchronization, the Sagnac experiment (Sagnac, 1913), and the gravitational frequency shift experiment. In order to discuss these, we first review some definitions and concepts.

The gravitational field variables are the metric coefficients, $g_{\mu\nu}$, and the space-time line element is related to these variables by

$$ds^2 = -g_{\mu\nu}\, dx^\mu\, dx^\nu \tag{2.1}$$

The three dimensional line element is expressed in terms of the space metric coefficients as

$$dl^2 = \gamma_{ij}\, dx^i\, dx^j \tag{2.2}$$

and the two sets of metric coefficients are related (Møller, 1955, p. 283) by

$$\gamma_{ij} = g_{ij} - \gamma_i \gamma_j, \qquad \Gamma_{ij} = \gamma_{ij}/(-g_{00}) \tag{2.3}$$

where

$$\gamma_i = g_{0i}/(-g_{00})^{1/2}, \qquad \Gamma_i = \gamma_i/(-g_{00})^{1/2} \tag{2.4}$$

The primary device for relating times on clocks which are separated in space is a light signal sent between them. The equation for a light signal, $ds = 0$, provides a quadratic equation for the time-of-flight of the signal, dx^0, whose solution (Landau & Lifshitz, 1962, p. 272) is:

$$dx_\pm{}^0 = [\Gamma_{ij}(a)\, dx^i\, dx^j]^{1/2} \pm \Gamma_i(a)\, dx^i \tag{2.5}$$

where (Fig. 2)

$$dx_-{}^0 = ct_B(p, a') \qquad \text{and} \qquad dx_+{}^0 = ct_B(a', q) \tag{2.6}$$

The light signal leaves particle B at p, is reflected at a on A, and returns to B at q; the event a' on B occurs at the same (global) time as event a on clock A.

In the transitivity of synchronization experiment, the event c on clock C is synchronized with b on B, which is synchronized with a on A (Fig. 3). One then measures the time difference between the events c and d on C, where d is synchronous with a on A. From equations (2.5) and (2.6) we get

$$ct_B(p, q) = dx_+{}^0 + dx_-{}^0 = 2ct_B(p, b) \tag{2.7a}$$

$$\tfrac{1}{2}ct_B(p, q) = [\Gamma_{ij}(a)\, dx^i_{AB}\, dx^j_{AB}]^{1/2} \tag{2.7b}$$

$$ct_B(a', b) = ct_B(p, b) - ct_B(p, a') = -\Gamma_i(a)\, dx^i_{AB} \tag{2.8a}$$

$$= ct_C(a'', b') \tag{2.8b}$$

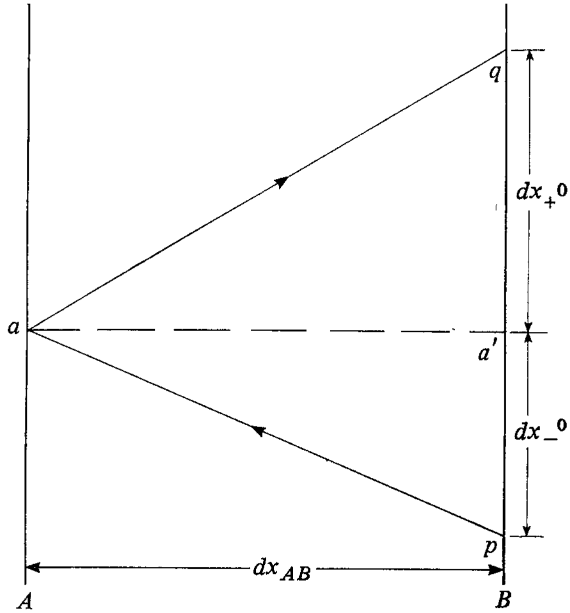

Figure 2—Time-of-flight of light signals.

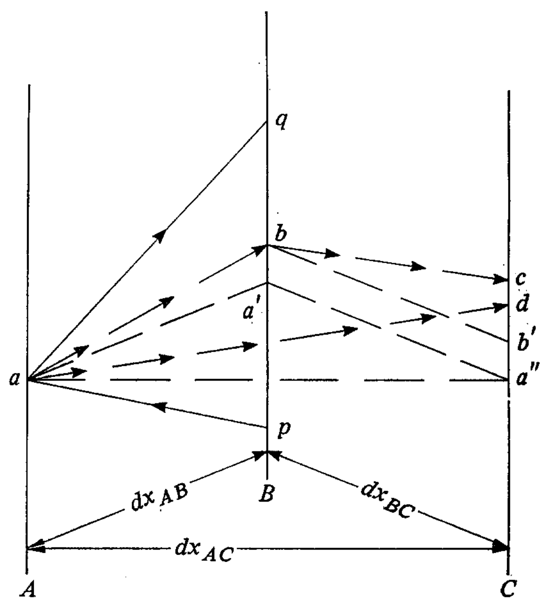

Figure 3—Transitivity of synchronization.

Similarly,

$$ct_C(b', c) = -\Gamma_i(b)\, dx^i_{BC} \tag{2.9}$$

$$ct_C(a'', d) = -\Gamma_i(a)\, dx^i_{AC} \tag{2.10}$$

The events a, a', and a'' all occur at the same global time, and so do b and b'. With the help of the relation

$$dx^i_{AC} = dx^i_{AB} + dx^i_{BC} \tag{2.11}$$

Equation (2.10) can be rewritten in the form

$$\begin{aligned} ct_C(a'', d) = -\Gamma_i(a)\, dx^i_{AB} + [\Gamma_i(a') - \Gamma_i(a) + \Gamma_i(b) - \\ -\Gamma_i(a')]\, dx^i_{BC} - \Gamma_i(b)\, dx^i_{BC} \end{aligned} \tag{2.12}$$

where the additional terms add up to zero.

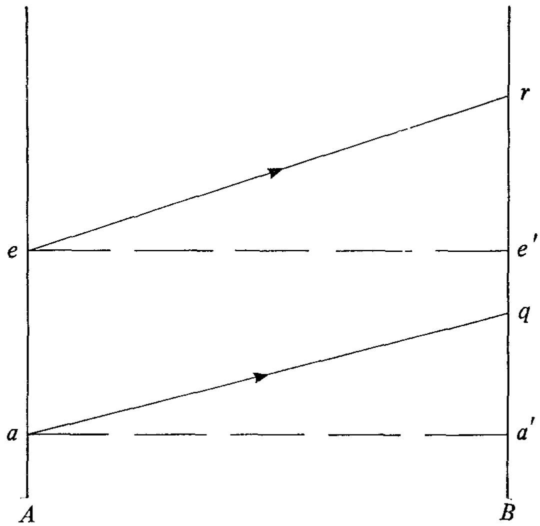

Figure 4—Derivatives, and frequency shift.

For an arbitrary function f we define its space and time partial derivatives (Fig. 4) as

$$f_{,i}\, dx^i_{AB} = f(a') - f(a) \tag{2.13a}$$

and

$$f_{,0}\, ct_A(a, e) = f(e) - f(a) \tag{2.13b}$$

respectively, where a and e are neighboring events on the same clock A, and a' on B occurs at the same time as a. Using this definition, we may rewrite equation (2.12), with the help of equations (2.8a–b) and (2.9), as

$$\begin{aligned} ct_C(a'', d) = ct_C(a'', b') + \Gamma_{i,j}(a)\, dx^i_{BC}\, dx^j_{AB} \\ -\Gamma_{i,0}(b)\, dx^i_{BC}\, ct_B(b, a') + ct_C(b', c) \end{aligned} \tag{2.14}$$

Bringing the first and last terms on the right side of equation (2.14) over to the left, we get

$$ct_C(c,d) = \Gamma_{i,j}(a)\,dx^i_{BC}\,dx^j_{AB} - \Gamma_{i,0}(b)\,dx^i_{BC}\,ct_B(b,a') \tag{2.15}$$

The interval $t_C(c,d)$ is that which is measured in the transitivity of synchronization experiment.

The symmetry of synchronization experiment involves only two particles: event c on clock A is synchronized with b on B which is synchronized with a on A, and one measures the time between the events a and c. The transitivity of synchronization experiment reduces to the symmetry of synchronization

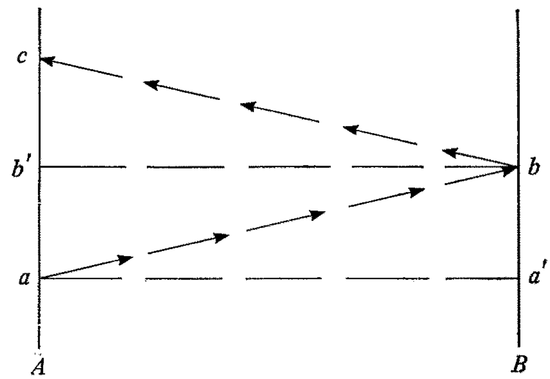

Figure 5—Symmetry of synchronization.

experiment if we let particles A and C coincide, in which case equation (2.11) becomes $dx^i_{BC} = -dx^i_{AB}$ and d and a become the same event (Fig. 5). With these considerations, equation (2.15) becomes

$$ct_A(c,a) = \Gamma_{i,0}(b)\,dx^i_{AB}\,ct_B(b,a') - \Gamma_{i,j}(a)\,dx^i_{AB}\,dx^j_{AB} \tag{2.16}$$

A similar experiment which involves a larger number of particles is the Sagnac experiment, which measures the difference in arrival times of two light signals sent around a closed path in opposite directions. The expression for the time difference, as measured by standard clocks, derived from equations (2.5) and (2.6) (Basri, 1965), is

$$c\Delta t = 2\int (\gamma_{i,j} - \gamma_{j,i})\,df^{ij} \tag{2.17}$$

where df^{ij} is an area element on the $x^i x^j$-surface ($i \neq j$).

The gravitational frequency shift experiment requires that two light signals be sent from the same point, one after the other (Fig. 4). If the two signals leave A at events a and e, and arrive at B at events q and r, respectively, then the difference in arrival times of the two signals is

$$t_B(q,r) = t_B(a',e') + t_B(e',r) - t_B(a',q) \tag{2.18}$$

We may substitute equations (2.5) and (2.6) for the last two terms of equation (2.18). Moreover, because a, a' and e, e' are at the same global time, $t_B(a',e') = t_A(a,e)$. If we consider this time difference to be infinitesimal, then we may apply our definition of the time partial derivative, equation (2.13b), and the result is

$$t_B(q,r) = t_A(a,e)\{1 + \Gamma_{i,0}(a)\,dx^i + [(\Gamma_{ij}(a)\,dx^i\,dx^j)^{1/2}]_{,0}\} \tag{2.19}$$

This can be rearranged slightly to give

$$\Gamma_{i,0}(a)\,dx^i = \frac{t_B(q,r)}{t_A(a,e)} - 1 - \{[\Gamma_{ij}(a)\,dx^i\,dx^j]^{1/2}\}_{,0} \tag{2.20}$$

The intervals $t_A(a,e)$ and $t_B(q,r)$ are measured by means of clocks A and B. Furthermore, g_{00} can be measured by comparing clock A to a coincident standard clock, and γ_{ij} is also measurable, either by using a length-measuring instrument and equation (2.2), or by a clock and equation (2.7a–b). Thus everything on the right-hand side of equation (2.20) is measurable; this gives us a means of measuring the time derivative of γ_i. Similarly, equations (2.15), (2.16), and (2.17) can be used to calculate $\gamma_{i,j}$ in terms of measurable quantities. Thus experiments used to test the symmetry and transitivity of synchronization, and the Sagnac experiment, can be considered as different methods of measuring $\gamma_{i,j}$. The first method is perhaps the easiest because it only involves two particles.

We have discussed several experimental methods of measuring the derivatives of γ_i. Henceforce when we speak of measuring $\gamma_{i,\mu}$, these experiments are implied.

3. *The Initial Surface*

In order to operationally establish the initial surface on which we assign the initial time value to all clocks, we must have a method of determining the event a' on B which occurs at the same time as the event a on the neighboring A. Once this can be done, we can simply begin at some origin and proceed from point to neighboring point, radiating outward, determining the events which lie on the initial surface.

Because there are only six independent field equations for the ten unknowns $g_{\mu\nu}$, four additional coordinate conditions must be prescribed. Ordinarily, three of the four are taken to be $g_{0i} = 0$, but this does not specify the coordinate system operationally. To specify the spatial coordinate system, we must prescribe three conditions on the space metric γ_{ij}, and to specify the time coordinate we must prescribe a condition of g_{00}. Once the coordinate system is thus specified, we have used up our four coordinate conditions; the values of g_{0i} are then restricted by the field equations, and cannot be chosen arbitrarily.

While the space and time derivatives of γ_i (and hence of g_{0i}) are measurable, γ_i itself is not. In order to measure γ_i, one must be able to measure $dx_+{}^0$ in equation (2.5), and if that were possible, then simultaneity would be absolute rather than relative as is assumed in relativity theory. That is, relativity requires that any event between p and q (Fig. 2) on B may be considered simultaneous with event a on A; but if we could measure $dx_+{}^0$, then we could determine *the one event* a' on B which is simultaneous with a. One would in effect be measuring the one-way speed of light, which would determine a coordinate system from which, by a Lorentz transformation, one could obtain a preferred inertial coordinate system, in which the one-way speed of light is the same as c, the round-trip speed.

It follows from the above that the value of γ_i is arbitrary to within an additive constant which cannot be measured. The range of possible values for this constant depends on whether it is required that causality hold for neighboring clocks. This point is discussed in more detail below. If causality is not required, then the constant can be assigned any finite real number. Given event a on A, the event a' on B is arbitrary, determined by the choice of the three constants for $\gamma_i(a)$, which in turn determine $dx_\pm{}^0$ on B, by equations (2.5) and (2.6). The event a'' on C, which occurs at the same time as a and a', is not arbitrary (Fig. 3), however; since the space derivatives are measurable, the relation

$$\gamma_i(a') = \gamma_i(a) + \gamma_{i,j}(a)\, dx^j_{AB} \tag{3.1}$$

determines $dx_+{}^0$ on C, again through equation (2.5), which determines the event a''.

We are now in a position to enumerate the procedures required to establish the initial surface operationally, and to secure all the necessary initial data on it. First we must choose a starting point, an origin, say at event a on particle A. At this point we must choose three values for the three γ_i. If it is required that the metric reduce to that of special relativity in an inertial frame, these three values must be taken to be zero. Three of the six γ_{ij} and g_{00} are fixed by the four coordinate conditions; we must measure the remaining three γ_{ij}, and repeat the measurements a short time later, to obtain the time derivatives of these quantities. The space and time derivatives of γ_i are then measured; this completes the data at a. Using our chosen values of γ_i in equation (2.5), we can calculate (not measure) values for $dx_+{}^0$, from which we may locate the events a', on neighboring particles, which occur at the same time as a. The value of γ_i at the neighboring particles is calculated from equation (3.1), and we simply begin all over again, radiating outward from the origin along the initial surface, obtaining the initial data as we go along.

This process of establishing the initial surface is unique once we have chosen the three values of γ_i at the origin. Nothing else is arbitrary, all is determined by the gravitational field.

4. *How a Unique Solution is Obtained*

If we consider the field equations at the initial surface, the curvature tensor may be written as (Adler, *et al.*, 1965, p. 212)

$$\begin{aligned} R_{ij} &= \tfrac{1}{2}g^{00}g_{ij,00} + M_{ij} \\ R_{0j} &= -\tfrac{1}{2}g^{0i}g_{ij,00} + M_{0j} \\ R_{00} &= \tfrac{1}{2}g^{ij}g_{ij,00} + M_{00} \end{aligned} \tag{4.1}$$

where $M_{\mu\nu}$ involves only the metric coefficients and their first time derivatives on the initial surface. If the coefficients are known over the entire initial surface, then clearly all orders of space derivatives are known as well. Equations (4.1) are ten in number, involving six unknowns $g_{ij,00}$. Thus the ten field equations can be separated into two parts: six equations of time evolution, and four equations amounting to compatibility requirements on the initial data (Adler, *et al.*, 1965, p. 215). These four relations can be used to reduce the number of measurements discussed at the end of the previous section. If we are measuring all the initial data, the compatibility requirements are automatically satisfied, and the six equations of time evolution enable us to compute the six metric coefficients g_{ij} for all time. Values for g_{00} and three of the γ_{ij} are determined by the coordinate conditions, and g_{0i} can be obtained from equation (2.3) in terms of g_{ij} and γ_{ij}. Thus we see that once the coordinate system is chosen and the initial surface is determined, the field equations yield a unique solution over all space-time.

If sources are involved, we must measure the energy-momentum tensor $T_\mu{}^\nu$ on the initial surface. If the sources are discrete and uncharged, the continuity equation (Møller, 1955),

$$T_\mu{}^\nu{}_{;\nu} = 0 \tag{4.2}$$

(the semicolon indicates covariant differentiation), or equivalently, the equation of motion, is sufficient to determine the location of the sources for all time. Additional equations are necessary whenever non-gravitational forces are involved. For instance, if electrical charges are present, Maxwell's equations are also needed. In the case of a fluid, we write the energy-momentum tensor in terms of the thermodynamic parameters of the fluid. Then, for example, a virial expansion [see, for example, Condon & Odishaw (1958)], relating these parameters to one another might comprise the further relations needed (i.e., an equation of state for the fluid). The virial coefficients in the expansion can be obtained from laboratory experiments on the fluid.

5. *Causality*

Ordinarily it is assumed that the arrival time of a light signal must be later than its departure time. This assumption would imply $dx^0_\pm > 0$ (Fig. 2). From this inequality and equation (2.5) we then conclude that

$$(\gamma_{ij}\,dx^i\,dx^j)^{1/2} > \gamma_i\,dx^i \tag{5.1}$$

Squaring and collecting terms, we get

$$(\gamma_{ij} - \gamma_i \gamma_j) dx^i dx^j > 0 \tag{5.2}$$

This inequality is true for arbitrary dx^i; a necessary and sufficient condition for this to be true is that all the subdeterminants of the array $(\gamma_{ij} - \gamma_i \gamma_j)$ be positive (Møller, 1955, p. 235) i.e.,

$$\begin{vmatrix} \gamma_{ii} & \gamma_{ij} & \gamma_{ik} \\ \gamma_{ij} & \gamma_{jj} & \gamma_{jk} \\ \gamma_{ik} & \gamma_{jk} & \gamma_{kk} \end{vmatrix} > \begin{vmatrix} (\gamma_i)^2 & \gamma_i \gamma_j & \gamma_i \gamma_k \\ \gamma_i \gamma_j & (\gamma_j)^2 & \gamma_j \gamma_k \\ \gamma_i \gamma_k & \gamma_j \gamma_k & (\gamma_k)^2 \end{vmatrix}$$

$$\begin{vmatrix} \gamma_{ii} & \gamma_{ij} \\ \gamma_{ij} & \gamma_{jj} \end{vmatrix} > \begin{vmatrix} (\gamma_i)^2 & \gamma_i \gamma_j \\ \gamma_i \gamma_j & (\gamma_j)^2 \end{vmatrix}, \quad \text{and} \quad \gamma_{ii} > \gamma_i^2 \tag{5.3}$$

($i \neq j \neq k \neq i$; summation over repeated indices is suppressed, this equation only). If it is desired that causality hold, then (5.3) must be considered as additional constraints on the solution. But since we already have enough conditions on the problem to ensure a unique solution, it may not be possible to satisfy equation (5.3) for all problems. In the case of the spherically symmetric field, for example, the solution using coordinate clocks for which

$$g_{00} = -1 + 2GM/rc^2$$

satisfies equation (5.3) everywhere (since γ_i is identically zero in that case). The solution to the same problem using standard clocks yields [Basri, 1965, equations (11.16), (13.14), (13.18), (13.21) and (13.22)]

$$\gamma_{11} = (1 - 2GM^2/rc^2)^{-1}$$

and

$$\gamma_1 = GMx^0[r^2 c^2(1 - 2GM/rc^2)]^{-1}$$

which satisfies (5.3) only in a finite region of space-time. Thus it seems that the choice of g_{00} might in some cases be the deciding factor in whether or not causality holds. There is unfortunately no relation derivable from equation (5.3) which might give us the general conditions on g_{00} necessary to ensure causality.

On the other hand, causality for clocks may actually be an unnecessary requirement (Landau & Lifshitz, 1962, footnote, p. 273). The equation of motion always predicts that any material particle arrives later than a light signal. Indeed, no non-physical predictions result from not requiring causality, merely a greater latitude in the available choice of coordinate clocks. In this case, equation (5.3) is disregarded.

6. *Conclusion*

Having specified the spatial coordinate system and the coordinate clocks, we have shown how to construct an initial time hypersurface which is consistent with the conditions imposed by any particular physical situation.

This construction is unique, aside from three additive constants in the values of γ_i. These constants can be set to zero by requiring the metric to reduce to that of special relativity in an inertial frame of reference.

Causality imposes additional conditions which may not be possible to satisfy in every situation. However, this causes no difficulties in the prediction of physical results.

This paper and the work of Basri (1965) complete the clarification of the operational foundation of Einstein's general theory of relativity.

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